MATM – MATEMATIKA

PER PARTES A SUBSTITUCE - ŘEŠENÍ ÚLOH

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Úlohy 1. (per partes)

1. $\int x\cos x\,\mathrm{d}x$

Řešení.

$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\cdot 1\,\mathrm{d}x = x\sin x - (-\cos x) + c = \\ = \underline{\underline{x\sin x + \cos x + c}}$$

$$\boxed{\begin{array}{ll} u = \underline{x} & v = \sin x \\ u' = 1 & v' = \underline{\cos x} \end{array}}$$

2. $\int x\mathrm{e}^x\,\mathrm{d}x$

Řešení.

$$\int x\mathrm{e}^x\,\mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x\cdot 1\,\mathrm{d}x = \underline{\underline{x\mathrm{e}^x - \mathrm{e}^x + c}}$$

$$\boxed{\begin{array}{ll} u = \underline{x} & v = \mathrm{e}^x \\ u' = 1 & v' = \underline{\mathrm{e}^x} \end{array}}$$

3. $\int x^2\sin x\,\mathrm{d}x$

Řešení. Per partes použijeme **dvakrát**!!!

$$\int x^2\sin x\,\mathrm{d}x = x^2(-\cos x) - \int -\cos x\cdot 2x\,\mathrm{d}x = \\ = -x^2\cos x + 2\underline{\int x\cos x\,\mathrm{d}x} =$$

(integrál znovu pomocí per partes dole)

$$= -x^2\cos x + 2\left(\underline{x\sin x + \cos x}\right) + c = \\ = \underline{\underline{-x^2\cos x + 2x\sin x + 2\cos x + c}}$$

$$\boxed{\begin{array}{ll} u = \underline{x^2} & v = -\cos x \\ u' = 2x & v' = \underline{\sin x} \end{array}}$$

$\underline{\int x\cos x\,\mathrm{d}x} = x\sin x - \int \sin x\cdot 1\,\mathrm{d}x = \underline{x\sin x + \cos x} + c$

$$\boxed{\begin{array}{ll} u = \underline{x} & v = \sin x \\ u' = 1 & v' = \underline{\cos x} \end{array}}$$

4. $\int x^2 \cos x \,\mathrm{d}x$

Řešení. Per partes použijeme **dvakrát**!!!

$$\int x^2 \cos x \,\mathrm{d}x = x^2 \sin x - \int \sin x \cdot 2x \,\mathrm{d}x =$$
$$= x^2 \sin x - 2\underline{\int x \sin x \,\mathrm{d}x} =$$

(integrál znovu pomocí per partes dole)

$$= x^2 \sin x - 2\left(\underline{-x\cos x + \sin x}\right) + c =$$
$$= \underline{\underline{x^2 \sin x + 2x\cos x - 2\sin x + c}}$$

$u = \underline{x^2}$	$v = \sin x$
$u' = 2x$	$v' = \underline{\cos x}$

$\underline{\int x \sin x \,\mathrm{d}x} = x(-\cos x) - \int -\cos x \cdot 1 \,\mathrm{d}x = \underline{-x\cos x + \sin x} + c$

$u = \underline{x}$	$v = -\cos x$
$u' = 1$	$v' = \underline{\sin x}$

5. $\int x^2 \mathrm{e}^x \,\mathrm{d}x$

Řešení. Per partes použijeme **dvakrát**!!!

$$\int x^2 \mathrm{e}^x \,\mathrm{d}x = x^2 \mathrm{e}^x - \int \mathrm{e}^x \cdot 2x \,\mathrm{d}x =$$
$$= x^2 \mathrm{e}^x - 2\underline{\int x \mathrm{e}^x \,\mathrm{d}x} =$$

(integrál znovu pomocí per partes dole)

$$= x^2 \mathrm{e}^x - 2\left(\underline{x\mathrm{e}^x - \mathrm{e}^x}\right) + c =$$
$$= \underline{\underline{x^2 \mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + c}}$$

$u = \underline{x^2}$	$v = \mathrm{e}^x$
$u' = 2x$	$v' = \underline{\mathrm{e}^x}$

$\underline{\int x e^x \, dx} = x e^x - \int e^x \cdot 1 \, dx = \underline{x e^x - e^x} + c$

$u = \underline{x}$	$v = e^x$
$u' = 1$	$v' = \underline{e^x}$

6. $\int \ln x \, dx$
Řešení.

$$\int 1 \cdot \ln x \, dx = \ln x \cdot x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 = \underline{\underline{x \ln x - x + c}}$$

$u = \underline{\ln x}$	$v = x$
$u' = \frac{1}{x}$	$v' = \underline{1}$

7. $\int \operatorname{arctg} x \, dx$
Řešení.

$$\int 1 \cdot \operatorname{arctg} x \, dx = \operatorname{arctg} x \cdot x - \int x \cdot \frac{1}{1+x^2} \, dx = x \operatorname{arctg} x - \int \frac{x}{1+x^2} \, dx =$$
$$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = \underline{\underline{x \operatorname{arctg} x - \frac{1}{2} \ln |x^2+1| + c}}$$

$u = \underline{\operatorname{arctg} x}$	$v = x$
$u' = \frac{1}{1+x^2}$	$v' = \underline{1}$

8. $\int x \ln x \, dx$
Řešení.

$$\int x \ln x \, dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x \, dx =$$
$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + c = \underline{\underline{\ln x \cdot \frac{x^2}{2} - \frac{x^2}{4} + c}}$$

$u = \underline{\ln x}$	$v = \frac{x^2}{2}$
$u' = \frac{1}{x}$	$v' = \underline{x}$

Úlohy 2. (substituce)

1. $\int \frac{\sin^3 x}{\cos x+3}\,\mathrm{d}x$

Řešení.

$$\begin{aligned}\int \frac{\sin^3 x}{\cos x+3}\,\mathrm{d}x &= \int \frac{\sin^3 x}{t+3}\cdot\frac{\mathrm{d}t}{-\sin x} = \int -\frac{\sin^2 x}{t+3}\mathrm{d}t = \int -\frac{1-\cos^2 x}{t+3}\mathrm{d}t = \\ &= \int -\frac{1-t^2}{t+3}\mathrm{d}t = \int \frac{t^2-1}{t+3}\mathrm{d}t = \text{ dělíme } (t^2-1):(t+3) = \\ &= \int t-3+\frac{8}{t+3}\mathrm{d}t = \int t\mathrm{d}t - \int 3\mathrm{d}t + \int \frac{8}{t+3}\mathrm{d}t = \\ &= \frac{t^2}{2} - 3t + 8\ln|t+3| + c = \\ &= \underline{\underline{\frac{\cos^2 x}{2} - 3\cos x + 8\ln|\cos x+3| + c}}\end{aligned}$$

$$\boxed{\begin{aligned} t &= \cos x \\ 1\cdot\mathrm{d}t &= -\sin x\cdot\,\mathrm{d}x \\ \mathrm{d}x &= \frac{\mathrm{d}t}{-\sin x}\end{aligned}}$$

2. $\int \frac{2x}{\sqrt{x+16}-4}\,\mathrm{d}x$

Řešení.

$$\begin{aligned}\int \frac{2x}{\sqrt{x+16}-4}\,\mathrm{d}x &= \int \frac{2\boxed{x}}{t-4}\cdot 2t\mathrm{d}t = \int \frac{2(t^2-16)}{t-4}\cdot 2t\mathrm{d}t = \\ &= 4\int \frac{t^3-16t}{t-4}\mathrm{d}t = \text{ dělíme } (t^3-16t):(t-4) = \\ &= 4\int t^2+4t\mathrm{d}t = 4\left(\frac{t^3}{3}+4\cdot\frac{t^2}{2}\right)+c = \\ &= \underline{\underline{4\left(\frac{\left(\sqrt{x+16}\right)^3}{3}+2\cdot\left(\sqrt{x+16}\right)^2\right)+c}}\end{aligned}$$

$$\boxed{\begin{aligned} t &= \sqrt{x+16} \\ t^2 &= x+16 \;\Rightarrow\; x = t^2-16 \\ 2t\cdot \mathrm{d}t &= 1\cdot \mathrm{d}x \\ \mathrm{d}x &= 2t\mathrm{d}t \end{aligned}}$$

3. $\int x^2(x^3+7)^{12}\,\mathrm{d}x$

Řešení.

$$\int x^2(x^3+7)^{12}\,\mathrm{d}x = \int x^2t^{12}\cdot\frac{\mathrm{d}t}{3x^2} = \frac{1}{3}\int t^{12}\mathrm{d}t = \frac{1}{3}\cdot\frac{t^{13}}{13}+c = \underline{\underline{\frac{(x^3+7)^{13}}{39}+c}}$$

$$\boxed{\begin{aligned} t &= x^3+7 \\ 1\cdot\mathrm{d}t &= 3x^2\cdot\mathrm{d}x \\ \mathrm{d}x &= \frac{\mathrm{d}t}{3x^2} \end{aligned}}$$

4. $\int 3x\cos(x^2-6)\,\mathrm{d}x$

Řešení.

$$\int 3x\cos(x^2-6)\,\mathrm{d}x = \int 3x\cos t\cdot\frac{\mathrm{d}t}{2x} = \frac{3}{2}\int\cos t\mathrm{d}t = \frac{3}{2}\sin t + c =$$
$$= \underline{\underline{\frac{3}{2}\sin(x^2-6)+c}}$$

$$\boxed{\begin{aligned} t &= x^2-6 \\ 1\cdot\mathrm{d}t &= 2x\cdot\mathrm{d}x \\ \mathrm{d}x &= \frac{\mathrm{d}t}{2x} \end{aligned}}$$